NACA TM 1385

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1385

ON THE GAS DYNAMICS OF A ROTATING IMPELLER

By A. Busemann

Translation of "Zur Gasdynamik des drehenden Schaufelsterns."
Zeitschrift für angewandte Mathematik und Mechanik,
vol. 18, issue 1, Feb. 1938.

Washington
March 1956

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1385

ON THE GAS DYNAMICS OF A ROTATING IMPELLER*

By A. Busemann

INTRODUCTION

Centrifugal pumps and turbines may sometimes be treated by using plane incompressible flow where the fluid springs from a source or runs into a sink in the center of the impeller. However, application of conformal mapping is not so simple for this type of flow as for rectilinear flow, because a steady flow about the blade sections exists only in a coordinate system rotating with the blade. In a coordinate system at rest, the velocity field outside of the singularities is free from sources and rotation, but the impeller rotates in it. By applying appropriate distributions of sources, sinks, and vortices inside the blade contour, a flow can be found which does not pass through the rotating blade contour. The action of the flow on these singularities inside the blades creates the torque.

For blowers with high circumferential velocities, the same considerations would have to be extended by the requirements of a compressible flow. Though there exist corresponding laws of forces on sources, sinks, and vortices, the conditions become considerably more involved because the freedom from vortices in the regular domain of the flow concerns the velocity field whereas freedom from sources exists only in steady flow and involves the stream-density field (the product of velocity and gas density). It becomes impossible to superimpose two fields free from sources and vortices into a new field which is free from sources and vortices; furthermore, in a compressible flow one can operate with distributed singularities only, since a certain region around point sources and point vortices is void of any velocity field. For these reasons, the laws of forces on singularities have only a very limited range of application in gas dynamics.

It is known that the velocity fields of incompressible flow and magnetic fields are similar with respect to the distribution of the vector and to the field energy. Accordingly, one should anticipate the forces on corresponding singularities to be equal so that a displacement of the singularities produces the amount of work required for the change in field energy connected with it. Contrary to this expectation the

*"Zur Gasdynamik des drehenden Schaufelsterns." Zeitschrift für angewandte Mathematik und Mechanik, vol. 18, issue 1, Feb. 1938, pp. 31-38, dedicated to the memory of the late editor Erich Trefftz.

hydrodynamic and the magnetic forces are always opposite in sign. The result is that, on the one hand, the forces on magnetic sources and sinks (north and south poles) and those on hydrodynamic vortices are such that their displacement work balances the field energy. On the other hand, for the displacement of hydrodynamic sources and sinks and of magnetic vortices (around conductors of electric currents), the opposite sign of the displacement work would be required to balance the field energy. The difference is restored for the magnetic vortex according to Faraday's induction law by the well-known fact that the electric current flowing in the conductor is opposed, during any displacement, by induced electromotive forces, the overcoming of which requires a supply of electrical energy. Likewise, additional energy must be supplied during displacements of the hydrodynamic sources and sinks; in this case the energy supply is caused by higher pressures at the location of the sources and lower pressures at the location of the sinks compared to the pressures of the steady flow. These pressure differences are precisely what constitutes the pressure heads or pressure differentials for rotating machinery.

For plane flow, one may regard instead of the field of the streamlines the field of the potential lines which are orthogonal to the streamlines. In the case of incompressible flow, this field is in the regular domain free from sources and vortices; however, sources and vortices are interchanged, since a streamline source represents a potential-line vortex and vice versa. Comparing plane potential lines with the plane field of magnetic lines of force, one finds now perfect agreement including the sign of the forces and the induction law. If one wants to determine the additional pressure difference between two points of the field which alters Bernoulli's pressure difference of the steady flow, one must draw a line connecting these points and observe on it the variation with time in the number of the potential lines which intersect this connecting line. The connecting line must not pass over source points of the potential lines. This is the same rule which applies for the determination of induced electromotive forces. In order to ascertain whether the connecting line has moved over a source, these sources cannot be allowed to appear and disappear. For magnetic sources, this prerequisite is obviously satisfied by the fact that a north pole can be created only by separating a north and a south pole of equal strength. The sources of the potential lines represent vortices of the streamlines and it follows from Helmholtz' vortex theorems that, in two-dimensional flow, a vortex rotating clockwise can be produced only by separating a vortex rotating clockwise from another one rotating counterclockwise. For these plane fields, the similarity is therefore very far reaching. For magnetic and hydrodynamic fields in space, the similarity is subject to limitations because lines of forces remain lines in space whereas the potential lines change to potential surfaces insofar as unique surfaces orthogonal to the streamlines exist. Generally, one is therefore limited to correlate only the hydrodynamic velocity field and the magnetic force field. If one identifies velocity and magnetic-field intensity, then the source matches the magnetic pole and the hydrodynamic vortex matches the wire through which current flows; with respect to the induction law, however, it is just the wire with electric current and the hydrodynamic source which can be compared.

Whereas the laws about forces on sources, sinks, and vortices almost maintain their form in transfer from hydrodynamics to gas dynamics but lose their major field of application, the hydrodynamical induction law must correspond to a similarly formulated gas-dynamical induction law and at the same time maintain to some extent its applicability, since it deals only with the pressure difference of unsteady flow as compared to steady flow. It is just this pressure difference which is related to the pressure rise in blowers. Thus, the author will give below the general derivation of the gas-dynamical induction law as he presented it for the first time in the summer of 1936 in a colloquium on gas dynamics at the DVL under the chairmanship of E. Trefftz. In a second part, he will show that the torque at the impeller in nonviscous compressible flow, even for circumferential velocities below sonic velocity, does not result from the effective pressure rise alone as in nonviscous incompressible flow, but that, on the contrary, even without effective pressure rise, energy quantities may be radiated from the rotating impeller.

I. THE PRESSURE RISE OF A CENTRIFUGAL BLOWER

1. Derivation of the Gas-Dynamical Induction Law

In the revision of the section "Hydrodynamics" for the 8th edition of A. Föppl, "Vorlesungen über technische Mechanik" (Lectures on technical mechanics), volume IV, page 417, I derived the "hydrodynamical induction law." If one limits the application to a loss-free gas flow of constant entropy, the derivation may be transferred directly to gases.

For a nonviscous gas, free from gravity, of pressure p, density ρ, and velocity W with the components u, v, w in the directions of the spatial coordinates x, y, z, one obtains, in dependence on the time t, the following equations of motion:

$$-\frac{\partial p}{\partial x} = \rho\frac{du}{dt}, \qquad -\frac{\partial p}{\partial y} = \rho\frac{dv}{dt}, \qquad -\frac{\partial p}{\partial z} = \rho\frac{dw}{dt} \tag{1}$$

For constant entropy s there applies for the enthalpy i of the gas:

$$di = \frac{dp}{\rho} \tag{2}$$

By substitution of this relationship into equation (1) there result the following equations:

$$-\frac{\partial i}{\partial x} = \frac{du}{dt}, \qquad -\frac{\partial i}{\partial y} = \frac{dv}{dt}, \qquad -\frac{\partial i}{\partial z} = \frac{dw}{dt} \tag{3}$$

For a certain time $t = t_o$ one can combine these three partial derivatives into the following total differential of the enthalpy in space:

$$-di = \frac{du}{dt}\,dx + \frac{dv}{dt}\,dy + \frac{dw}{dt}\,dz \tag{4}$$

If one introduces the convention that the components dx, dy, dz of a line element in space (which are used in equation (4) only for the time $t = t_o$) shall be for all times, the components of a line element attached to the gas which connects the adjacent points G_1 and G_2 moved with the gas, the following transformation of equation (4) holds:

$$-di = \frac{d}{dt}\,(u\,dx + v\,dy + w\,dz) - u\,du - v\,dv - w\,dw \tag{5}$$

since in this case the interchange of the differentiation

$\frac{d}{dt}\,(dx) = d\left(\frac{dx}{dt}\right) = du$, etc., is valid. If one places the points G_1 and G_2 attached to the gas farther apart, one may integrate the differentials indicated in equation (5) along a line from G_1 to G_2 moved with the gas, so that the following difference in enthalpy is then obtained:

$$i_1 - i_2 = \frac{d}{dt}\left[\int_{G_1}^{G_2} (u\,dx + v\,dy + w\,dz)\right] + \frac{1}{2}\left(u_1^2 + v_1^2 + w_1^2\right) - \frac{1}{2}\left(u_2^2 + v_2^2 + w_2^2\right) \tag{6}$$

If one introduces, instead of the points G_1 and G_2 attached to the gas, the points fixed in space P_1 and P_2, which at the time $t = t_o$ coincide with the former, one can prolong the line between G_1 and G_2

with those gas points sweeping over the points P_1 and P_2 after the time $t = t_0$ (fig. 1). The integration along this line, which is likewise attached to the gas (and is therefore determined in the development with time) but connects the fixed points P_1 and P_2, includes beyond the integral required in equation (6) contributions which result from the shift of the integration limits and have the value $\vec{W}_1^2 - \vec{W}_2^2$ dt. If one subtracts the latter, one obtains the following relation:

$$i_1 - i_2 = \frac{d}{dt}\left[\int_{P_1}^{P_2} (u\,dx + v\,dy + w\,dz)\right] - \frac{1}{2}(u_1^2 + v_1^2 + w_1^2) +$$

$$\frac{1}{2}(u_2^2 + v_2^2 + w_2^2)$$

or

$$i_2 + \frac{1}{2}(u_2^2 + v_2^2 + w_2^2) - \left[i_1 + \frac{1}{2}(u_1^2 + v_1^2 + w_1^2)\right] =$$

$$-\frac{d}{dt}\int_{P_1}^{P_2} (u\,dx + v\,dy + w\,dz) \tag{7}$$

The right side of the equation represents the induced pressure rise which is generated between the points P_1 and P_2. Note that one deals here not with a partial differentiation with respect to time but with a total differentiation along a line moved with the gas. The selection of the line at the time t_0 is arbitrary. If this line, once selected, is not moved exactly with the gas, there originates an error which is proportional to the vortices located between the line moved exactly with the gas and the wrong connecting line.

2. Applications of the Induction Law

In a steady gas flow free from vortices, the right side of equation (7) disappears because in the first place the integral, due to the freedom from vorticity, is independent of the path and equals the potential difference $\Phi_2 - \Phi_1$ and in the second place, this potential difference, due to the steady state, is independent of the time.

For a steady gas flow with vortices, the right side of equation (7) disappears only for points P_1 and P_2 which lie on the same streamline. If one draws the connecting line from P_1 to P_2 along this streamline, all its points remain on this streamline later on, too. The entire integration path varies with time only in that it extends beyond the downstream point P_2 in a flat loop which makes, however, no contribution to the integral. The value of the integral on the remaining piece from P_1 to P_2 is again independent of the time so that the right side of equation (7) disappears. The vanishing right side of equation (7) establishes on the left side the validity of the Bernoulli equation for the gas flow.

For a flow which is free from vortices but variable with time, the value of the integral is at every instant, independently of the path, equal to the potential difference $\Phi_2 - \Phi_1$. However, since the velocity distribution varies with time, the potential difference also becomes dependent on the time, and one obtains the well-known relationship:

$$i_2 + \frac{1}{2}\left(u_2^2 + v_2^2 + w_2^2\right) - \left[i_1 + \frac{1}{2}\left(u_1^2 + v_1^2 + w_1^2\right)\right] = -\frac{\partial}{\partial t}\left(\Phi_2 - \Phi_1\right)$$

or

$$i + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{\partial\Phi}{\partial t} = \text{Constant} \tag{8}$$

This is the generalized Bernoulli equation for the unsteady gas flow free from vortices.

A further application of the induction law is possible for nonsteady gas flows which vary periodically with time. In this case, it is easier to determine, instead of the pressure rise at every instant, the mean value of the pressure rise for the period of the duration T. If one integrates the right side of equation (7) with respect to t over the period and then divides by the duration of this period, one obtains the time average:

$$h = \frac{1}{T}\left\{\left[\int_{P_1}^{P_2} (u\,dx + v\,dy + w\,dz)\right]_{t_o} + \left[\int_{P_2}^{P_1} (u\,dx + v\,dy + w\,dz)\right]_{t_o+T}\right\} \tag{9}$$

Since the velocity field is the same at the times t_o and $t_o + T$, the mean pressure rise indicated above represents the circulation in one instantaneous field on a closed line from P_1 via P_2 back to P_1 divided by the time T.

Figure 2 shows how to apply this rule to the periodic flow in a rotor. One recognizes that the circulation required by equation (9) matches exactly the circulation Γ around the blade. In the case of m blades and an angular velocity of the impeller ω, the period is given by $T = \frac{2\pi}{m\omega}$; accordingly, the pressure rise of the rotor amounts to:

$$h = \frac{\Gamma}{T} = \frac{\Gamma m\omega}{2\pi} \tag{9a}$$

Thus, the significance of the circulation around the blades of the impeller with respect to the pressure rise is established for all gas flows without increase in entropy. However, it remains to be investigated whether also the torque and therewith the power consumption depends in the same manner on the blade circulation.

II. TORQUE OF THE IMPELLER WITHOUT MASS FLOW

The torque of the impeller in plane flow may be determined from the difference of the moments of momentum over a control circle outside the impeller and one inside the impeller. If one forms in polar coordinates the velocity components in the direction of the radius w_r and of the circumference w_u, with w_u counted positive in the direction of the increasing angle ψ, there results on the arc element $r\,d\psi$ of a circle of the radius r the mass flow $\rho w_r r\,d\psi$. The momentum of this quantity in circumferential direction is obtained by multiplication by w_u, the moment of momentum by multiplication by rw_u. By integration of the moment of momentum over the entire circumference one obtains a torque

$$D = r^2 \int_0^{2\pi} \rho w_r w_u \, d\psi \tag{10}$$

If the entering fluid in the interior of the impeller is supplied without rotation, a sufficiently small radius may exist on which circumferential components of the velocity do not yet appear. In this case, the integral over the external circle according to equation (10) already yields the torque of the impeller. For incompressible flow with $\rho = \text{const.}$, the velocity components w_r and w_u do not correlate in the absence of obstacles outside of the impeller, so that for the integration over w_u the mean value of w_r may be taken out from under the integral sign. The mean value of w_r multiplied by the density ρ and the circumference of the circle $2r\pi$, however, is simply the mass G discharged per second through the impeller. Therefore, in the case of incompressible flow, the torque may be split into the following factors:

$$D = \frac{G}{2\pi}\int_0^{2\pi} w_u r \, d\psi = Gm\frac{\Gamma}{2\pi} = \frac{Gh}{\omega} \tag{11}$$

Here m is the number of blades, and Γ the circulation around each of the m blades; the latter transformation results from equation (9a). Thus, for an incompressible flow a discharge as well as a circulation around the blades is necessary when a torque is to occur.

Since, according to the results of the previous section, the pressure rise also depends directly on the blade circulation for compressible flow, there would also be a certain justification for assuming that in the case of compressible flow no correlation between discharge and blade circulation enters equation (10). However, one single example where a torque occurs without discharge or without blade circulation will be sufficient for deciding this question in the negative sense. In the case of the example selected, there appears neither a discharge nor a blade circulation, and yet one obtains at higher velocities a torque different from zero.

In order to represent the flow for the case of vanishing discharge and vanishing blade circulation at a large distance from the impeller, one can replace the impeller by a rotating wavy cylinder. The simplest wavy cylinder has a cross section in which on the circumference of a circle of the radius R, m sinusoidal waves with the wave amplitude A are superimposed (fig. 3 for the case $m = 3$). In case one wants to represent the effect of an impeller with m blades more accurately, one could still add further waves with the amplitudes A_2, A_3, etc., and the numbers $2m$, $3m$, etc. If the amplitude A, for the sake of further simplification of the calculation, is limited to small values compared to the radius R or, more specifically, compared to the wave length $L = \frac{2R\pi}{m}$, one can neglect in equation (8) the square of the gas velocity compared to the other summands. Considering constant entropy according to equation (2) one then obtains

$$i + \frac{\partial\Phi}{\partial t} = \int \frac{dp}{\rho} + \frac{\partial\Phi}{\partial t} = \text{Constant} \tag{12}$$

From the potential $\Phi(r,\psi)$ the velocity components are obtained in the following manner:

$$w_r = \frac{\partial\Phi}{\partial r} \qquad w_u = \frac{1}{r}\frac{\partial\Phi}{\partial\psi} \tag{13}$$

Expressed by these components the continuity equation of the unsteady gas flow reads:

$$\frac{\partial\rho}{\partial t} + \frac{\rho}{r}\left[\frac{\partial(rw_r)}{\partial r} + \frac{\partial w_u}{\partial\psi}\right] = \frac{\partial\rho}{\partial t} + \rho\left[\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\psi^2}\right] = 0 \tag{14}$$

From equations (12) and (14) there results the well-known equation of sound propagation:

$$\Delta\Phi \equiv \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\psi^2} = \frac{1}{a^2}\frac{\partial^2\Phi}{\partial t^2} \tag{15}$$

in which a is the sonic velocity of the gas according to the following relationship:

$$a^2 = \frac{dp}{d\rho} \tag{16}$$

Since, for small velocities, the pressure and the density deviate only little from the values of the gas at rest p_o and ρ_o, a may be regarded as constant and equal to the value of the sonic velocity for this state. By integration of the pressure at constant density, there results from equation (12) the pressure

$$p = p_o - \rho_o\frac{\partial\Phi}{\partial t} \tag{17}$$

The impenetrability of the surface of the wavy cylinder furnishes the boundary condition for the differential equation (15). At the time $t = 0$ the cylinder has the following radii r depending on the central angle ψ:

$$r = R - A \sin m\psi \tag{18}$$

Due to the rotation at the angular velocity ω (fig. 3), one obtains the dependency of the cylinder radii on ψ and t

$$r(\psi,t) = R - A \sin m(\psi - \omega t) \tag{19}$$

Hence, the spinning wavy cylinder produces a radial pumping motion with the velocity

$$w_r = \frac{\partial r}{\partial t} = Am\omega \cos m(\psi - \omega t)$$

Because of the smallness of the amplitude A, it is sufficient to prescribe this value for the radial component of the gas velocity. According to equation (13), there results the boundary condition for Φ:

$$w_r = \left(\frac{\partial \Phi}{\partial r}\right)_{r=R} = Am\omega \cos m(\psi - \omega t) \tag{20}$$

Since the integration proper of the differential equation for the propagation of sound is known and we are here concerned only with the application to the impeller, the detailed calculation and the determination of the integration constants may be omitted after stating that the elimination of the constants was performed in such a manner that in the asymptotic development of the solution for large radii only outgoing waves (no incoming waves) were retained. The solution which thereby became unique may be written in terms of the Bessel functions of the first and second kind J_m and Y_m and reads

$$\Phi = aA \frac{Y_m\left(\frac{m\omega r}{a}\right) \cos (m\psi - m\omega t + \delta) + J_m\left(\frac{m\omega r}{a}\right) \sin (m\psi - m\omega t + \delta)}{\sqrt{\left[Y_m'\left(\frac{m\omega R}{a}\right)\right]^2 + \left[J_m'\left(\frac{m\omega R}{a}\right)\right]^2}} \tag{21}$$

The phase angle δ in equation (21), though in itself unessential, has the value

$$\tan \delta = \frac{J_m'\left(\frac{m\omega R}{a}\right)}{Y_m'\left(\frac{m\omega R}{a}\right)} \tag{22}$$

If one substitutes this solution into the equation (10) for the torque, its integration can be achieved with the aid of the formula[1]:

$$J_m(x)Y_m'(x) - Y_m(x)J_m'(x) = \frac{2}{\pi x}$$

One obtains thus the torque

$$D = \frac{2\rho_o m a^2 A^2}{\left[J_m'\left(\frac{m\omega R}{a}\right)\right]^2 + \left[Y_m'\left(\frac{m\omega R}{a}\right)\right]^2} \tag{23}$$

The power consumption $E = D\omega$ corresponds to the radiated sound output.

In order to represent the results in dimensionless form, we shall use as the Mach number M of the gas flow the ratio of the circumferential velocity u and the sonic velocity a:

$$M = \frac{u}{a} = \frac{R\omega}{a} \tag{24}$$

A coefficient for the resistance to the motion c_w equal to the torque D divided by the dynamic pressure $q = \frac{1}{2}\rho_o u^2$, the generating surface of the cylinder $F = 2R\pi$, and the radius R is introduced:

$$c_w = \frac{D}{\frac{1}{2}\rho_o u^2 2R^2\pi} \tag{25}$$

---

[1]Compare Frank-Mises: Differentialgleichungen der Physik (Differential equations in physics), 1930, vol. 1, p. 414.

and a second coefficient is formed for the radiated energy c_e equal to the radiated energy $E = D\omega$ divided by q, F, and the sonic velocity a:

$$c_e = \frac{E}{\frac{1}{2}\rho_0 u^2 a 2R\pi} = c_w M \tag{26}$$

In this form the results of the calculation are plotted in figures 4 and 5 with the Mach number used as the abscissa and the coefficients c_w or c_e used as the ordinate. The values m indicated at the individual curves signify the number of waves on the circumference of the circle. The ordinate has as its unit a value which is formed from the amplitude A and the wave length L, or the amplitude A, the number of waves m, and the radius R.

One recognizes from the figures that the torque and likewise the radiated energy disappears only for incompressible flow and for gas flow with very small velocities. In the subsonic domain there results for growing m an ever increasing region in which no noteworthy torques occur. The maximum of c_w always lies at the Mach number 1.

It is interesting to compare the rotating wavy cylinder in a resting gas and the resting wavy cylinder with a circulatory flow treated by G. J. Taylor[2]. While it was found there that even at velocities higher than sonic velocity damped perturbation waves occur, the above calculation, inversely, yields the result that even below sonic velocity the disturbance extends to infinity. Both cases agree at $m = \infty$ (that is, for a cylinder radius large compared to the wave length) with the solution for a flat plate with sinusoidal waves treated before by J. Ackeret[3].

SUMMARY

It is shown by the example of the plane impeller that in a gas flow with constant entropy as in an incompressible flow, the pressure head of a rotating impeller depends only on the circulation around the blades. In contrast to incompressible flow, however, one obtains for the impeller rotating freely in an infinite gas mass a larger torque than would be necessary for production of the pressure rise because, due to the periodic disturbance caused by the rotating impeller, sound

---

[2]G. I. J. Taylor: ZAMM 10, 1930, p. 334.

[3]J. Ackeret: Helvetia Phys. Acta 1, 301, 1928.

waves of finite energy travel away to infinity. This energy produces a resistance which grows with a high initial power of the circumferential velocities and, when sonic velocity is exceeded, gradually becomes the wave resistance of bodies moved rectilinearly[3] at supersonic velocity.

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics

---

[3]See footnote 3 on p. 12.

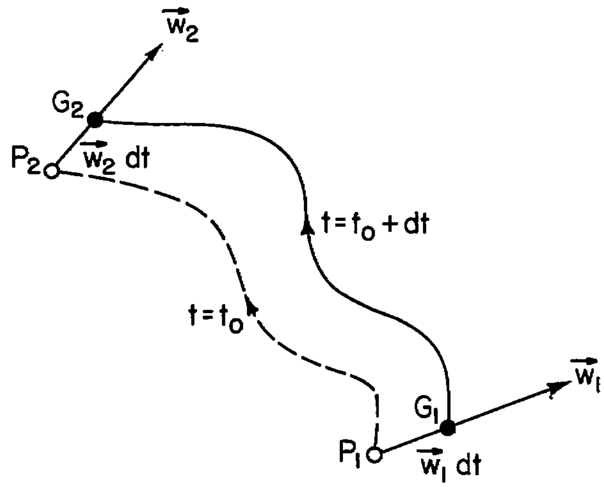

Figure 1.- Moving path of integration.

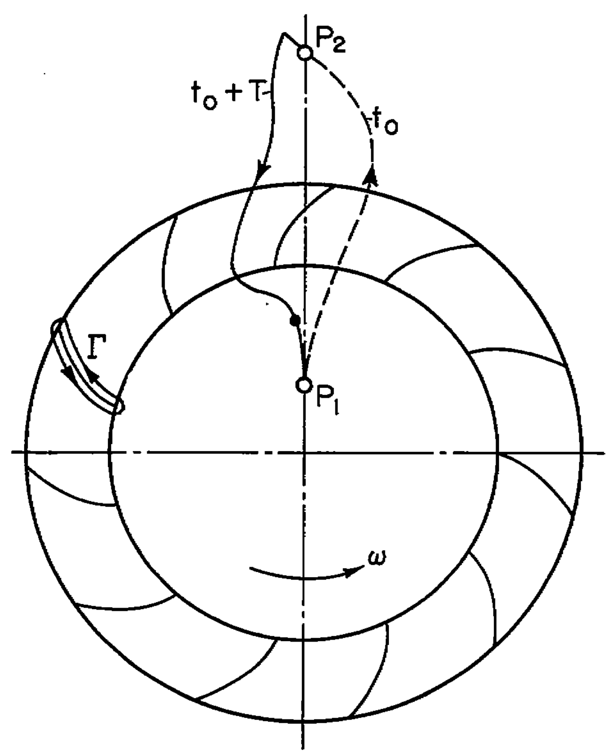

Figure 2.- Rotating impeller.

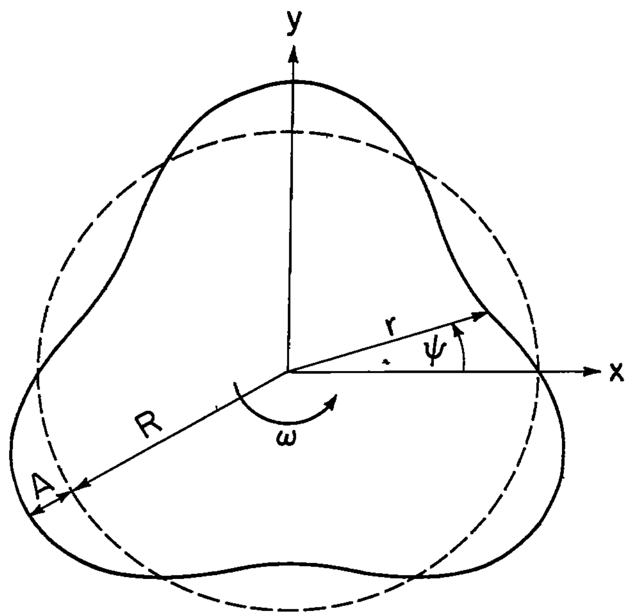

Figure 3.- Cross section of the cylinder corrugated by $m = 3$ waves.

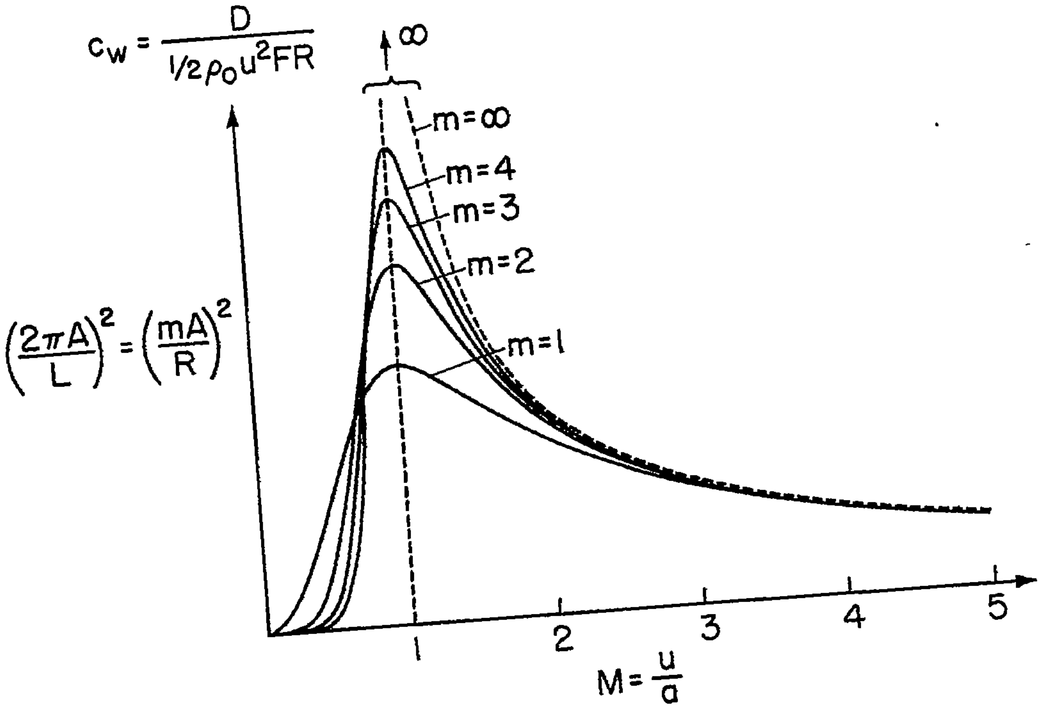

Figure 4.- Resistance for rotation of the wavy cylinder.

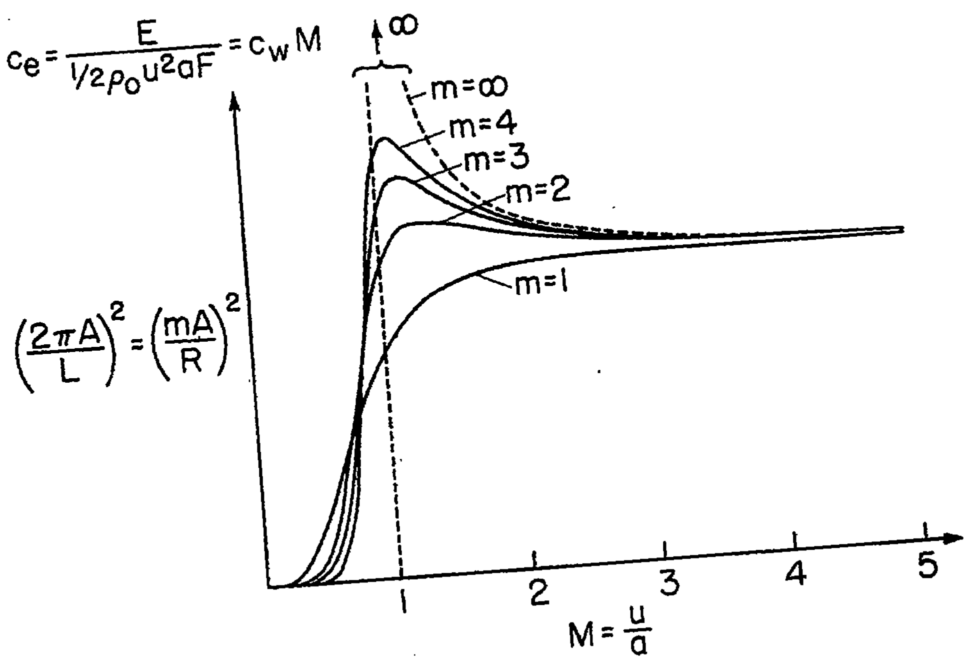

Figure 5.- Energy radiation for rotation of the wavy cylinder.

NACA - Langley Field, Va.